Math 10A
Worksheet, Discussion #26; Monday, 7/23/2018
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1 Recurrence Relations

1.1 Concepts

1. In order to solve a linear homogeneous we can replace the equation with its characteristic polynomial. For instance, the characteristic polynomial of $a_n = 2a_{n-1} + a_{n-2}$ is $\lambda^2 = 2\lambda + 1$. Then if $\lambda_1, \dots, \lambda_k$ are roots of this polynomial, then the general form of the solution is $a_n = C_1\lambda_1^n + \cdots + C_k\lambda_k^n$.

1.2 Problems

2. **TRUE** False If a_n, b_n are two solutions to a linear homogeneous equation, then $a_n + b_n$ is also an solution.

> **Solution:** If our recurrence was $a_n = 2a_{n-1}$ for instance, then $(a_n + b_n) = (2a_{n-1} + 2b_{n-1}) = 2(a_{n-1} + b_{n-1})$ showing that their sum is also a solution.

3. **TRUE** False If a_n is a solution to a linear homogeneous equation, then ca_n is also a solution for any constant c.

4. Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_1 = 5, a_2 = 13$.

> **Solution:** This is a homogeneous equation and so we can solve this using the characteristic equation. The characteristic polynomial is $\lambda^2 = 5\lambda - 6$ or $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$. Thus, the general form of the solution has $a_n = c_1 2^n + c_2 3^n$. Now to solve for these constants, we plug in our initial conditions. We have that $a_1 = 2c_1 + 3c_2 = 5$ and $a_2 = 4c_1 + 9c_2 = 13$. Solving gives us $c_1 = c_2 = 1$ so the solution is $a_n = 2^n + 3^n$.

5. Find a recurrence relation such that $a_n = 2^n - 3^n$ is a solution to it. (Hint: What would the characteristic polynomial be?)

> **Solution:** Since the bases of the exponentials are 2 and 3, the characteristic polynomial would be $(\lambda - 2)(\lambda - 3) = \lambda^2 - 5\lambda + 6$. So, the recurrence relation would be $a_n - 5a_{n-1} + 6a_{n-2} = 0$ or $a_n = 5a_{n-1} - 6a_{n-2}$.

6. Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$.

Solution: The characteristic polynomial is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$. Thus the general form is $a_n = c_1 4^n + c_2(-1)^n$. Plugging in our initial conditions gives $c_1 + c_2 = 3$ and $4c_1 - c_2 = 2$ which gives $c_1 = 1$ and $c_2 = 2$. So the answer is $a_n = 4^n - 2 \cdot (-1)^n$.

7. Find a recurrence relation such that $a_n = n(-1)^n$ is a solution to it.

Solution: Since we have $n(-1)^n$, we know that there is a double root of $\lambda = -1$ so the characteristic polynomial is $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$. So the recurrence relation is $a_n + 2a_{n-1} + a_{n-2} = 0$ or $a_n = -2a_{n-1} - a_{n-2}$.

8. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 4$.

Solution: The characteristic polynomial is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)(\lambda - 2) = 0$. Thus the general form is $a_n = c_1 2^n + c_2 n 2^n$. Plugging in our initial conditions gives $c_1 = 3$ and $2c_1 + 2c_2 = 4$ which gives $c_1 = 3$ and $c_2 = -1$. So the answer is $a_n = 3 \cdot 2^n - n2^n$.